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Infiltration from supply at constant water content: an integrable model

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Abstract An integrable version of Richards' equation for time-dependent unidimensional flow in unsaturated soil is subjected to boundary conditions of constant water content. The nonlinear boundary problem is transformed to a linear diffusion problem with modified Stefan boundary conditions. A formal series is developed, leading to successive approximations to the solution at early times. Each additional term of the series for the location of the free boundary in the transformed problem leads directly to another coefficient in the Philip infiltration series in the original problem.

Keywords Infiltration · Integrable model · Kummer functions · Series methods · Similarity solutions

1 Introduction

Throughout the 20th Century, understanding of flow in porous media improved greatly through concomitant developments in the theory of nonlinear parabolic equations in conservation-law form. Such equations arise in soil-water continuum models that predict concentrations averaged over sampling regions that are large compared to individual pores and grains. The continuum models have withstood the test of time. Many applied scientists have found them to be quite adequate to predict macroscopic flow properties at field and laboratory scales, in applications such as soil-water flow, oil recovery and industrial filtration.

Following Buckingham's modification of Darcy's law to account for unsaturated soil, volumetric water flux density is given by

$$\mathbf{J}(\mathbf{r}, t) = K(\theta)\mathbf{k} - D(\theta)\nabla\theta(\mathbf{r}, t), \tag{1.1}$$

where θ is the volumetric water content, t is time, z is depth below the soil surface in the vertical direction $\mathbf{k}$, $D(\theta)$ is the soil-water diffusivity and $K(\theta)$ is the hydraulic conductivity. From the conservation of water mass, transport in unsaturated soil is governed by the Richards equation,

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$$\frac{\partial \theta}{\partial t} = \nabla \cdot [D(\theta)\nabla\theta] - \frac{\mathrm{d}K(\theta)}{\mathrm{d}\theta}\frac{\partial \theta}{\partial z}. \tag{1.2}$$

This is a minor modification of the equation of Richards [1] who used pressure rather than water content as the dependent variable.

Nonlinear Fokker–Plank equations have many applications. In the applications of water flow in unsaturated soil and of two-phase oil–water flow, nonlinearity is very important since $K(\theta)$ and $D(\theta)$ are found to be strongly increasing concave functions. For the history, physical basis and modelling principles behind Eq. (1.2), the reader is referred to [2, Sect. 9.4.6], [3, Chaps. 9–11], [4, Chap. 7], [5]. For up-to-date accounts of numerical, exact and approximate analytic solution methods, see for example [6, Chap. 1], [7, Chaps. 3–4], [8, Chap. 5].

The convective term in (1.2) would not be present without gravity (e.g. [5]). It is often found in practice that gravity is not important for some time after the commencement of water inflow (e.g. [5,9,10]). During these early times, the standard gravity-free porous medium equation is of some use. However, over the time scales of many weeks required for recharge to aquifers and for regional solute transport, gravity may be even more important than diffusion (e.g. [11]). In order to study the evolving interplay between gravity and diffusion, it is instructive to consider idealised but physically meaningful nonlinear boundary-value problems for vertical one dimensional flow, with $\frac{\partial}{\partial z}$ replacing ∇ in (1.2).

In the 1950s, Philip [5,12,13] began to build a general analytical approach to constructing the classical solution of the vertical infiltration problem on the half-line $[0, \infty)$, subject to uniform initial conditions and Dirichlet boundary conditions for constant water content:

Problem 1

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial z}\left(D(\theta)\frac{\partial \theta}{\partial z}\right) - \frac{\mathrm{d}K(\theta)}{\mathrm{d}\theta}\frac{\partial \theta}{\partial z} \quad (z,t) \in [0,\infty) \times [0,\infty),$$

$$\theta(z,0) = \theta_n \quad z \in (0,\infty), \quad \theta(0,t) = \theta_s \quad t \in (0,\infty), \tag{1.3}$$

$$\theta(z,t) \to \theta_n \quad \text{as} \quad z \to \infty,$$

where θ_n and θ_s denote constants.

Problem 1 may model rapid saturation, without ponding, at the surface of an initially dry soil. The depth at which the volumetric water content is θ, is given by a power series in $\sqrt{t}$,

$$z = \phi_0(\theta)t^{1/2} + \phi_1(\theta)t + \cdots \phi_j(\theta)t^{(1+j)/2} + \cdots \tag{1.4}$$

This leads to the Philip infiltration series for the equivalent depth $i(t)$ of liquid to have entered the soil,

$$i(t) - K_n t = S_0 t^{1/2} + S_1 t + \cdots S_j t^{(1+j)/2} + \cdots \tag{1.5}$$

For convenience of subsequent calculations, the constant $K_n = K(\theta_n)$ is separated from coefficient S_1.

Now $zt^{-1/2} = \phi_0(\theta)$ is the form of the well-known similarity solution to the gravity-absent nonlinear diffusion equation with Dirichlet boundary conditions. The function $\phi_0(\theta)$ is known explicitly [14–17] for several nonlinear diffusivity functions $D(\theta)$. For these exact solutions, the parameter S_0, known as the sorptivity [13] is calculated explicitly. However, when gravity is significant, the other functions $\phi_j(\theta)$ have so far been obtained only by numerically solving a sequence of integro–differential equations [5]. Some solutions have been obtained from special unrealistic simplified models with D constant, as in the linear and Burgers convection–diffusion equations, or by assuming a Green–Ampt step function water content profile, which would follow from a delta function diffusivity. However, the higher infiltration coefficients S_j have not been calculated exactly when $D(\theta)$ and $K(\theta)$ are fixed realistic functions and the initial water content θ_n is allowed to vary.

Beginning in the mid 1980s, several applications were found [18–24] for the integrable nonlinear convection–diffusion equation originating from [25], with

$$D(\theta) = \frac{a}{(b-\theta)^2}, \tag{1.6}$$

$$K(\theta) = \frac{\lambda}{2(b-\theta)} + \beta + \gamma(b-\theta), \tag{1.7}$$

with λ, β, γ, a and b positive constants. Most importantly, this model has provided an exact solution for flow in unsaturated soil subject to the boundary condition of constant flux,

$$K(\theta) - D(\theta)\frac{\partial\theta}{\partial z} = r \text{ (const)} \quad \text{at } z = 0. \tag{1.8}$$

This represents a realistic model for flow in unsaturated soil under conditions of steady irrigation or rainfall in the case $r > 0$ [21–24], steady evaporation in the case $r < 0$ [26] and drainage in the case $r = 0$ [27]. Broadbridge et al. [28] extended the solution to a finite domain with the constant-flux boundary condition augmented by a zero-flux barrier condition at some depth. Warrick et al. [29] solved this equation with piecewise-constant flux boundary conditions. Some exact solutions with special types of continuously variable flux boundary conditions were provided in [30].

The constant-flux boundary condition (1.8), a nonlinear Robin condition, has been amenable to exact solution even though Problem 1, with the Dirichlet condition, has defied attempts at exact solution for at least 20 years.

Section 2 develops an integrable model of the type (1.6–1.7). In Sect. 3, using the integrable model, Problem 1 is converted to the linear heat equation with a modified Stefan free-boundary condition.

Section 4 develops a series approach for the equivalent modified Stefan problem. This involves a power series in $\sqrt{t}$, which follows naturally after separating variables in a new coordinate system in which the pseudo-steady state is equivalent to a time-dependent similarity solution in the original variables. The coefficients of the power series can be expressed explicitly in terms of confluent hypergeometric functions.

2 An integrable nonlinear model

In order to illustrate a pathway to solve Problem 1, consider a slightly simplified integrable model. As before, this assumes a diffusivity function of the form (1.6). The parameters a and b of the nonlinear diffusivity function $D = a/(b-\theta)^2$ may be estimated from diffusion experiments, as explained by Broadbridge and White [22,23]. If S_0 is matched to the value of experimental sorptivity, then [22]

$$a = h(C)S_0^2, \tag{2.1}$$

where C is the nonlinearity parameter,

$$C = \frac{b-\theta_n}{\theta_s-\theta_n}, \tag{2.2}$$

θ_n and θ_s being respectively the antecedent water content and the water content at saturation, $\theta_n < \theta_s < b$. Here $h(C)$ is the solution of a familiar transcendental equation

$$\frac{1}{C} = \left(\frac{4h}{\pi}\right)^{-1/2} \mathrm{e}^{1/4h}\mathrm{erfc}\left(\frac{1}{2h^{1/2}}\right),$$
$$\frac{1}{2} < \frac{h(C)}{C(C-1)} < \frac{\pi}{4}. \tag{2.3}$$

Now assume a restricted conductivity function with $\gamma = 0$ in (1.7). That is,

$$K(\theta) = \frac{\lambda}{2(b-\theta)} + \beta. \tag{2.4}$$

The remaining two parameters λ and β may be fixed by matching the values of conductivity at $\theta = \theta_s$ and θ_n, implying

$$\lambda = 2C(C-1)(K_s-K_n)(\theta_s-\theta_n), \qquad \beta = K_n - (C-1)(K_s-K_n).$$

This guarantees that the solution has the correct travelling-wave speed at large times. It is well known [31] that at large t, the solution asymptotically approaches a travelling-wave profile with speed

$$U = \frac{K_s - K_n}{\theta_s - \theta_n}. \tag{2.5}$$

In the formulation of Broadbridge and White [22], the additional free parameter γ was specified by a somewhat artificial but usually harmless assumption that $\mathrm{d}K/\mathrm{d}\theta = 0$ at $\theta = \theta_n$. This was an approximation that ultimately simplified the exact solutions. A more general value of γ was allowed in the book by Smith et al. [7, Chap. 4]. With the present boundary conditions, the mathematics is simplified differently by setting the parameter γ to zero, at the expense of allowing a less realistic soil hydraulic pressure potential Ψ, given by

$$\frac{\mathrm{d}\Psi}{\mathrm{d}\theta} = \frac{D(\theta)}{K(\theta)}.$$

In this model, $D(\theta)$ and $K'(\theta)$ are proportional, a restriction that has been found to be approximately true in many practical situations [5,10].

As in Broadbridge and White [22], it is convenient to define rescaled water content, and dimensionless depth and time by

$$\Theta = \frac{\theta - \theta_n}{\theta_s - \theta_n}, \tag{2.6}$$

$$z_* = z/\ell_s; \qquad \ell_s = \frac{1}{K_s - K_n} \int_{\theta_n}^{\theta_s} D(\theta)\mathrm{d}\theta, \tag{2.7}$$

$$t_* = t/t_s; \qquad t_s = \frac{\ell_s(\theta_s - \theta_n)}{K_s - K_n} = \frac{h}{C(C-1)} \frac{S_0^2}{(K_s - K_n)^2}. \tag{2.8}$$

The depth scale ℓ_s is a conductivity-weighted average capillary rise, interpreted as a macroscopic sorptive length scale; t_s is a gravity time scale, a typical time taken for gravity to begin to dominate capillary action. It should be true that $S_1 t_s^{1/2}/S_0 = O(1)$.

In terms of these dimensionless quantities, the vertical flow equation is

$$\frac{\partial\Theta}{\partial t_*} = \frac{\partial}{\partial z_*}\left(\frac{C(C-1)}{(C-\Theta)^2}\frac{\partial\Theta}{\partial z_*} - \frac{C(C-1)}{(C-\Theta)}\right). \tag{2.9}$$

This is a conservation equation with renormalised flux, renormalised diffusivity and renormalised conductivity

$$\begin{aligned} J_* &= \frac{J - \beta}{K_s - K_n} = K_* - D_*\frac{\partial\Theta}{\partial z_*}, \\ D_* &= \frac{D t_s}{\ell_s^2} = \frac{C(C-1)}{(C-\Theta)^2}, \\ K_* &= \frac{K - \beta}{K_s - K_n} = \frac{C(C-1)}{(C-\Theta)}. \end{aligned} \tag{2.10}$$

Although it is possible to scale the parameter C out of the governing equation by a linear change of variables, this is not done here as C would then reappear in the initial conditions, and also the time and length units would need a new interpretation when C is close to 1. Note that in this model,

$$\begin{aligned} K_*'(\Theta) = D_*(\Theta) &\to 1 \text{ as } C \to \infty, \\ &\to \delta(\Theta - 1) \text{ as } C \to 1^+. \end{aligned}$$

If the function $D(\Theta)$ is extended outside the domain [0, 1] by defining $D(\Theta)$ to be zero for $\Theta > 1$ or $\Theta < 0$, then it can be shown that it approaches the Dirac delta function $\delta(\Theta - 1)$ as $C \to 1^+$. Hence, the nonlinear model resembles the linear model as C increases to large values, and it resembles a model of the Green–Ampt type, with delta function diffusivity, as C approaches 1. Now define

$$w = \frac{C - \Theta}{R} e^{z_*}, \quad y = 1 - e^{-z_*}, \tag{2.11}$$

with $R = [C(C-1)]^{1/2}$. Equation (2.9) is thus simplified to

$$\frac{\partial w}{\partial t_*} = \frac{\partial}{\partial y}\left(\frac{1}{w^2}\frac{\partial w}{\partial y}\right). \tag{2.12}$$

Consider a new integrated variable

$$u = \int_0^y w(y_1, t_*) dy_1 + S(t_*), \tag{2.13}$$

for some function S. Then

$$\frac{\partial u}{\partial t_*} = \left(\frac{\partial u}{\partial y}\right)^{-2} \frac{\partial^2 u}{\partial y^2} - \frac{J_*(0, t_*)}{R} + S'(t_*). \tag{2.14}$$

Therefore, a convenient choice is

$$S(t_*) = \frac{1}{R}\int_0^{t_*} J_*(0, t_1) dt_1, \tag{2.15}$$

so that

$$\frac{\partial u}{\partial t_*} = \left(\frac{\partial u}{\partial y}\right)^{-2} \frac{\partial^2 u}{\partial y^2}. \tag{2.16}$$

Note that $RS'(t_*)$ is the renormalised surface flux, leading to a linear relationship between cumulative infiltration rate and $S'(t_*)$. Cumulative infiltration is the total depth of water having entered the soil,

$i(t) = i_-(t) + K_n t$,

where $i_-(t) = \int_0^\infty (\theta(z, t) - \theta_n) dz$.

Then

$$i'_-(t) = (K_s - K_n)(R\, S'(t_*) + 1 - C). \tag{2.17}$$

Now applying the hodograph transformation, there emerges the classical heat equation [30],

$$y_{t_*} = y_{uu}. \tag{2.18}$$

The initial condition is

$$y_u(u, 0) = \kappa[1 - y], \tag{2.19}$$

where $\kappa = [(C-1)/C]^{1/2}$. The consistent boundary condition at infinity is

$$y(u, t_*) \to 1 \quad \text{as } u \to \infty. \tag{2.20}$$

However the constant water content at the soil surface implies a system of two boundary conditions at a free boundary $u = S(t_*)$:

$$\begin{aligned} &y(S(t_*), t_*) = 0 \\ &y_u(S(t_*), t_*) = \kappa^{-1}. \end{aligned} \tag{2.21}$$

In order to simplify the boundary conditions, define a new dependent variable

$$V = R[y_u + \kappa(y - 1)]. \tag{2.22}$$

Noting that

$$\frac{\mathrm{d}}{\mathrm{d}t_*}y(S(t_*),t_*) = \kappa^{-1}\frac{\mathrm{d}S(t_*)}{\mathrm{d}t_*} + y_{t_*}(S(t_*),t_*) = 0,$$

the free-boundary-value problem for $V(u,t_*)$ is

Problem 2

$$V_{t_*} = V_{uu} \tag{2.23}$$

$$S(0)=0,\ \ V(u,0)=0;\quad V(u,t)\to 0 \ \text{ as } u\to\infty,$$

$$V(S(t_*),t_*) = 1, \tag{2.24}$$

$$V_u(S(t_*),t_*) = R - C\frac{\mathrm{d}S(t_*)}{\mathrm{d}t_*}. \tag{2.25}$$

In the context of heat conduction this would be equivalent to the somewhat artificial problem of a melting solid absorbing latent heat, as sensible heat is conducted in from the neighbouring liquid but with an additional steady rate of heat supply at the boundary. An approximate large-t solution has been developed for a similar problem with $R<0$ and $C<0$, in the context of laser drilling, in which case some of the laser energy is conducted as heat into the solid [32, Chap. 6], [33, Chap. 4].

The location of the free boundary in Problem 2 leads directly to the cumulative infiltration in the original soil-water flow Problem 1, via Eq. 2.17. In the next section, a formal series representation is developed for early times.

3 Series approach for free-boundary problem

For the standard Stefan problem (Problem 2 with $R=0$), it is well known from the classical Lamé–Clapeyron–Neumann solution (e.g. [34, Sect. 1.3]) that there is a similarity solution $v=f(Y)$ with $Y=u/\sqrt{t_*}$, implying

$$S'(t_*) = \frac{1}{2}\gamma_0 t_*^{-1/2}, \tag{3.1}$$

with γ_0 constant. Since at early times, this singular function dominates any constant R in (2.25), assume that (3.1) still gives the leading order of the boundary location in a series,

$$S(t_*) = \gamma_0 t_*^{1/2} + \gamma_1 t_* + \gamma_2 t_*^{3/2} + \cdots \tag{3.2}$$

In Philip's infiltration theory [5], at sufficiently early times, the water-content profile is close to a similarity solution that is valid in the absence of gravity. It follows from that theory that in the current context, at times $t_*\ll 1$, the solution must be close to the leading-order approximation, which is the scale-invariant similarity solution of the standard Stefan problem. We suppose that

$$V \approx V_0(Y),$$

where $V_0(Y)$ is defined by

$$V_0(Y) = \int_Y^\infty \mathrm{e}^{-z^2/4}\mathrm{d}z \Big/ \int_{\gamma_0}^\infty \mathrm{e}^{-z^2/4}\ \mathrm{d}z = \mathrm{erfc}(Y/2)/\mathrm{erfc}(\gamma_0/2), \tag{3.3}$$

with γ_0 the solution of

$$\sqrt{\pi}\gamma_0 \mathrm{e}^{\gamma_0^2/4}\mathrm{erfc}(\gamma_0/2) = 2/C. \tag{3.4}$$

Comparing (2.3) and (3.4), it follows that

$$h(C) = \gamma_0^{-2}. \tag{3.5}$$

Since the leading-order approximate solution is scale invariant, it is convenient to change independent variables from (u, t_*) to (Y, T), where $Y = u/\sqrt{t_*}$ and $T = t_*$. In fact the canonical variables of the scaling symmetry (e.g. [35, Sect. 1.4.2]) are Y and $\log(T)$ but it is more instructive to retain the time coordinate. The heat equation (2.23) becomes

$$TV_T = \frac{1}{2}YV_Y + V_{YY}. \tag{3.6}$$

At the free boundary,

$$Y = \Sigma(T) = ST^{-1/2} = \gamma_0 + \gamma_1 T^{1/2} + \gamma_2 T + \cdots \tag{3.7}$$

The free-boundary conditions are

$$V(\Sigma(T), T) = 1, \tag{3.8}$$

$$T^{-1/2}V_Y(\Sigma(T), T) = R - C\left(\frac{1}{2}T^{-1/2}\Sigma(T) + T^{1/2}\frac{\mathrm{d}\Sigma(T)}{\mathrm{d}T}\right). \tag{3.9}$$

Importantly, Eq. 3.6 admits a very simple separation of variables $V(Y, T) = F(T)G(Y)$, with

$$TF'(T) = kF, \quad G''(Y) + \frac{1}{2}YG'(Y) = kG. \tag{3.10}$$

The relationship between symmetry and separation of variables for linear partial differential equations, has been discussed by Miller [36, Chap. 1]. In conformity with the power series in $\sqrt{T}$ for $S(T)$, it is consistent to choose $k = j/2$ to be a half integer, and

$$F_j(T) = T^{j/2}; \quad j = 0, 1, 2, \ldots$$

Now Eq. 3.10 is equivalent to Kummer's equation. One choice of independent basis solution that approaches zero at infinity is

$$G_j(Y) = \mathrm{e}^{-Y^2/4}\Psi\left((j+1)/2, 1/2, Y^2/4\right),$$

where Ψ is a standard confluent hypergeometric function, defined exactly as the Kummer U function of [37, Sect. 13.1]. This choice of solution satisfies $G(Y) \to 0$ as $Y \to \infty$, agreeing with boundary conditions $V \to 0$ as $u \to \infty$ and initial condition $V \to 0$ when $t_* \to 0$. Consider a power series expansion in $\sqrt{T}$,

$$V = V_0(Y) + \mathrm{e}^{-Y^2/4}\sum_{j=1}^{\infty} C_j T^{j/2}\Psi\left((j+1)/2, 1/2, Y^2/4\right). \tag{3.11}$$

For convenience, $V_0(Y)$ will sometimes be expressed as $C_0\mathrm{e}^{-Y^2/4}\Psi\left(1/2, 1/2, Y^2/4\right)$, where $C_0 = \left[\sqrt{\pi}\ \mathrm{erfc}(\gamma_0/2)\right]^{-1}$. It is a special property of this coordinate system (Y, T) that each term in the $\sqrt{T}$-power series in (3.11) is by itself a solution of the governing equation. In the original coordinate system (u, t_*) a power series of terms $t_*^k H_k(u)$ would not have this property because the separation of variables is no longer valid. Note that the leading term, a similarity solution, is a pseudo-steady state, not an actual steady state in the original coordinate system. Similar functions were used long ago by Gibson [38] and by Langford [39] to solve other Stefan problems. A closely related separation of variables, involving integral error functions of Y, was used by Tao [40] to solve classical Stefan problems with general time-dependent flux boundary conditions and general initial conditions.

Substituting the power-series expansion (3.11) in the modified Stefan Problem 2, the governing diffusion equation is already satisfied at all orders of $\sqrt{T}$ and so are the initial condition $V(Y, 0) = 0,\ Y > 0$ and the boundary condition at infinity, $V(Y, T) \to 0$. Substituting (3.7) in the free-boundary conditions (3.8) and (3.9), it follows that

$$\begin{aligned} 1 = V\left(\sum_{i=0}^{\infty}\gamma_i T^{i/2}, T\right) &= \sum_{k=0}^{\infty} C_k T^{k/2} G_k\left(\sum_{i=0}^{\infty}\gamma_i T^{i/2}\right) \\ &= \sum_{k=0}^{\infty} C_k T^{k/2}\left\{G_k(\gamma_0) + \sum_{j=1}^{\infty}\frac{1}{j!}G_k^{(j)}(\gamma_0)\left(\sum_{i=1}^{\infty}\gamma_i T^{i/2}\right)^j\right\} \end{aligned} \tag{3.12}$$

and

$$V_Y\left(\sum_{i=0}^{\infty}\gamma_i T^{i/2}, T\right) = R\,T^{1/2} - C\sum_{j=0}^{\infty}\frac{1}{2}(j+1)\gamma_j T^{j/2}. \tag{3.13}$$

In order to obtain the coefficients γ_j and C_j, V_Y in (3.13) is expanded by term-by-term differentiation of the formal series in (3.11). At the leading degree T^0, there is the similarity solution (3.3)–(3.4). By equating terms of order $T^{1/2}$, there follows a system of linear equations for C_1 and γ_1:

$$C_1 \mathrm{e}^{-\gamma_0^2/4}\Psi(1, 1/2, \gamma_0^2/4) + V_0'(\gamma_0)\gamma_1 = 0$$

$$C_1\gamma_0 \mathrm{e}^{-\gamma_0^2/4}\left[-\frac{1}{2}\Psi(1, 1/2, \gamma_0^2/4) \ + \frac{1}{2}\Psi'(1, 1/2, \gamma_0^2/4)\right] - \frac{1}{2}\gamma_1\gamma_0 V_0'(\gamma_0)$$
$$= R - C\gamma_1$$

Note that in the above, use has been made of the identity $V_0''(Y) = -\frac{1}{2}Y V_0'(Y)$. By using (3.5), the solution to the linear system is expressed as

$$\gamma_1 = \frac{4h\kappa\Psi\,(1, 1/2, 1/4h)}{\Psi'\,(1, 1/2, 1/4h) + 4h\Psi(1, 1/2, 1/4h)}, \tag{3.14}$$

$$C_1 = \frac{2h^{1/2}C\kappa e^{1/4h}}{\Psi'\,(1, 1/2, 1/4h) + 4h\Psi(1, 1/2, 1/4h)}. \tag{3.15}$$

In evaluating the derivative of the Kummer function, one useful identity is

$$\Psi'(a, b, z) = -a\Psi(a+1, b+1, z).$$

For any $j \geq 2$, the coefficients C_j and γ_j are determined from explicit recurrence relations given in the Appendix. The first correction to the solution of the free-boundary Problem 2, beyond the approximate similarity solution, is

$$V = \mathrm{erfc}(Y/2)/\mathrm{erfc}(\gamma_0/2) + t_*^{1/2} 2h^{1/2}[C(C-1)]^{1/2}\mathrm{e}^{(\gamma_0^2 - Y^2)/4}$$
$$\times\frac{1}{\Psi'(1, 1/2, 1/4h) + 4h\Psi(1, 1/2, 1/4h)}\Psi(1, 1/2, Y^2/4).$$

From the solution of Problem 2 for the linear diffusion equation, one may construct a parametric solution

$$u \to (z_*(u, t_*), \Theta(u, t_*))$$

to the nonlinear Fokker–Planck equation (2.9):

$$z_* = \kappa(u - S) - \log\left(1 - \frac{\mathrm{e}^{-\kappa S}}{R}\int_S^u V(u', t_*)\mathrm{e}^{\kappa u'}\mathrm{d}u'\right) \tag{3.16}$$

$$\Theta = \frac{C\mathrm{e}^{z_*}V}{C - 1 + \mathrm{e}^{z_*}V}. \tag{3.17}$$

When $V(u, t_*)$ is an arbitrary solution to the heat equation, the construction (3.16)–(3.17) will not be an exact solution of the nonlinear Fokker–Planck equation (2.9) unless (2.16) is recovered exactly, without additional functions of time on the right-hand side, as in (2.14). This is equivalent to $S(t_*)$ satisfying the differential equation

$$\frac{\mathrm{d}S}{\mathrm{d}t_*} = \kappa - \frac{V_u(S(t_*), t_*)}{V(S(t_*), t_*) + C - 1}. \tag{3.18}$$

When the series for $S/\sqrt{t_*}$ and $V(Y, t_*)$ are truncated at order $t_*^{j/2}$, the initial condition $\Theta = 0$ will be correct but the boundary value $\Theta(0, t_*)$ will be approximate at best. In the numerical examples depicted in Figs. 1 and 2, we achieved boundary values $\Theta(0, t_*)$ close to one by evaluating one to twenty additional terms in the series. This is a different situation from that of the Philip series solution [12], which maintains correct initial and boundary conditions at all levels but the governing equation is approximated for longer times as the number of terms increases.

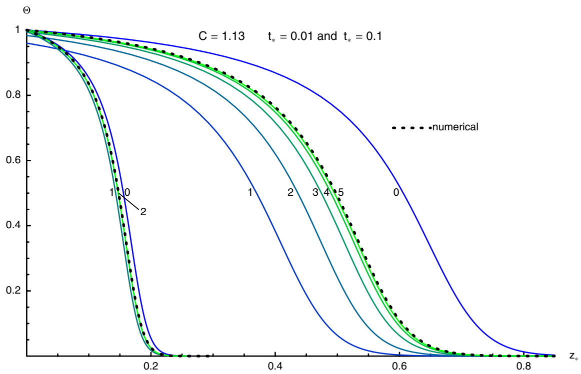

Fig. 1 Water-content profile $\Theta(z_*, t_*)$ versus z_* at dimensionless times $t_* = 0.01$ and $t_* = 0.1$ for a nonlinear model with $C = 1.13$. A curve labelled with integer n shows the solution based on an $n + 1$ term approximation of $V(u, t_*)$

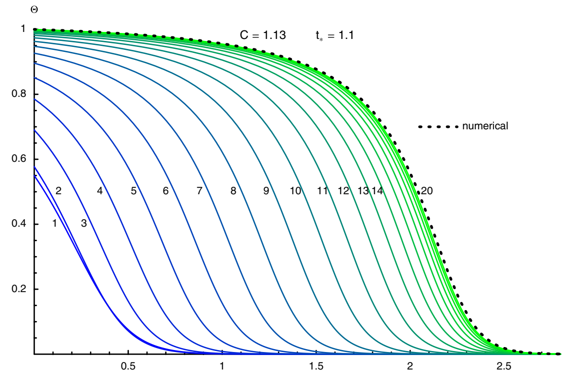

Fig. 2 Water-content profile at dimensionless time $t_* = 1.1$ for a nonlinear model with $C = 1.13$. A curve labelled with integer n shows the solution based on an $n + 1$ term approximation of $V(u, t_*)$

Even the first ($j = 0$) approximation is good for a short time because $V_0(Y)$ is a quasi-steady solution, embodying some realistic time dependence within the similarity variable $Y = u/\sqrt{t_*}$. At this lowest order of approximation, the expression relating z_* to u is

$$\begin{aligned}(C-1)\mathrm{erfc}(\gamma_0/2)&\left[\mathrm{e}^{-z_*}\mathrm{e}^{\kappa(u-\gamma_0\sqrt{t_*})}-1\right]\\ &= \mathrm{e}^{(\kappa^2 t_*-\kappa\gamma_0\sqrt{t_*})}\mathrm{erfc}\left(u/2\sqrt{t_*}-\kappa\sqrt{t_*}\right)-\mathrm{e}^{\kappa(u-\gamma_0\sqrt{t_*})}\mathrm{erfc}\left(u/2\sqrt{t_*}\right)\\ &\quad+\mathrm{erfc}(\gamma_0/2)-\mathrm{e}^{\kappa^2 t_*-\kappa\gamma_0\sqrt{t_*}}\mathrm{erfc}\left(\gamma_0/2-\kappa\sqrt{t_*}\right).\end{aligned} \tag{3.19}$$

Figure 1 displays the water content profile for the soil with $C = 1.13$, which is appropriate for the data of a light clay [23]. The bold dotted lines show water content profiles that result from direct numerical solution of (2.9) with appropriate boundary conditions. Curves labelled with an integer n show profiles reconstructed from an $n + 1$ term series approximation for $V(u, t_*)$, with the integral in (3.16) evaluated numerically. At the early time $t_* = 0.01$, (3.19) is a reasonable approximation to the actual water-content profile.

Figure 2 displays the corresponding outputs at time $t_* = 1.1$. To obtain such good agreement with the accurate numerical solution, 21 terms were included in the series construction. In Philip's series method [5], "practical convergence" was not observed for times much longer than this.

The series construction includes an evaluation of the higher infiltration coefficients. The explicit recurrence relations for the higher-order series and infiltration coefficients are given in the Appendix. The second infiltration coefficient is discussed in more detail in the next section.

4 Evaluation of infiltration coefficients

From (2.17),

$$i_-(t) = S_0 t^{1/2} + S_1 t + \cdots,$$

where

$$\begin{aligned}S_1 &= (K_s - K_n)(C-1)\left(\frac{4h\Psi(1, 1/2, 1/4h)}{\Psi'(1, 1/2, 1/4h) + 4h\Psi(1, 1/2, 1/4h)} - 1\right)\\ &= (K_s - K_n)(C-1)\left(\frac{\Psi(2, 3/2, 1/4h)}{4h\Psi(1, 1/2, 1/4h) - \Psi(2, 3/2, 1/4h)}\right).\end{aligned} \tag{4.1}$$

At the extreme of a linear soil model, $C \to \infty$, it is convenient to use the expansions

$$h(C) = \frac{\pi}{4}C^2 + O(C) \tag{4.2}$$

$$\Psi\,(1, 1/2, z) = 2 - 2\sqrt{\pi}z^{1/2} + O(z) \tag{4.3}$$

$$\Psi'(z) = -\sqrt{\pi}z^{-1/2} + O(1) \tag{4.4}$$

to deduce

$$S_1 \to \frac{1}{2}(K_s - K_n). \tag{4.5}$$

This agrees with a prediction made directly by solving a linear convection-diffusion model. The linear model is inadequate for many practical purposes. Its predicted water-content profiles do not have the observed inflection point and they do not have steep gradients that are observed near a wet front. Unlike models that have $K''(\theta)$ positive, the travelling-wave solution, also observed in experiments, is not stable for a linear model.

Since for many repacked soils, C is typically found to be in the range [1.02,1.5] [23], it is important to check the limiting behaviour of (4.1) as C approaches 1 from above, which is the limit of extremely strong dependence of hydrological transport coefficients on water content. As $C \to 1$ and $z = 1/(4h(C)) \to \infty$, one may use the expansions

$$h(C) = \frac{1}{2}[C-1] + O([C-1]^2), \tag{4.6}$$

$$\Psi(a, b, z) = z^{-a} + a(b-a-1)z^{-a-1} + O(z^{-a-2}) \tag{4.7}$$

to deduce

$$S_1 \to \frac{1}{3}(K_s - K_n), \quad \text{as } C \to 1. \tag{4.8}$$

This is in accord with experimental observations that S_1/K_s usually lies in the range 0.3 to 0.4 [41]. However, it disagrees with the direct prediction of the Green–Ampt model, $S_1/(K_s - K_n) = 2/3$. In fact, Fig. 1 of [41], shows that when the linear model is modified to a nonlinear model with increasing variability of $D(\theta)$, but with $D/K'(\theta)$ remaining constant, $S_1/(K_s - K_n)$ should continue to decrease below its initial value of 0.5. It is expected that this limiting value of $S_1/(K_s - K_n) = 1/3$ for an extremely variable diffusivity should be more realistic than the value of 2/3 predicted directly by the Green–Ampt model. A similar outcome occurred also in predicting time to incipient ponding during a uniform rainfall rate represented by the constant-flux boundary condition. The limiting prediction of time to ponding is different from and more realistic than that of the Green–Ampt model [21].

Philip [41] developed an inverse approach in which a feasible functional form was posed for $\phi_0(\Theta)$ and $\phi_1(\Theta)$ after which $D(\Theta)$ and $K'(\Theta)$ were deduced in the form of integrals that were not evaluated explicitly. This led to useful indicative relationships among the infiltration parameter S_1, the degree of nonlinearity of $D(\Theta)$ and the degree of nonlinearity of $K'(\Theta)$, lending support to the current predictions of S_1. However, in order to achieve manageable integral expressions for $D(\Theta)$ and $K'(\Theta)$, Philip adopted simple models in which the water content profiles do not have inflection points and for which $\phi_0(0)$ and $\phi_1(0)$ take finite values, which corresponds to a degenerate model for which both $D(\Theta)$ and $K'(\Theta)$ must be zero at the initial water content. In effect, this means that the inverse approach will not allow one to vary the initial water content while keeping the same soil hydraulic model $(K(\theta), D(\theta))$. The series construction, presented here, is not restricted in this way.

5 Conclusion

Much of the theory of infiltration has been based on Problem 1 for unidimensional flow in a semi-infinite unsaturated soil under Dirichlet conditions of constant water content at the boundary. The Philip series solution has

$$z = \sum_{j=0}^{\infty} \phi_j(\theta)t^{(j+1)/2}.$$

Unfortunately, however, to evaluate such expansions, one still needs to numerically solve a sequence of complicated nonlinear integro–differential equations for $\phi_j(\theta)$. In this paper, using an integrable model, Problem 1 has been transformed to a modified Stefan problem with linear heat conduction. Unlike the classical Stefan problem, the full Problem 2 is not invariant under a classical Lie symmetry group and it cannot be tackled by direct reduction of variables. However after adopting the canonical coordinates of the approximate Boltzmann scaling symmetry, the classical heat equation allows separation of variables, leading to a series of powers of T multiplied by Kummer functions of $Y^2/4$. Such a series is compatible with the modified Stefan problem using only half-integer powers of T.

Figure 2 evidences that at least up to dimensionless times around 1, increasing the number of terms in the series does lead to successive improvements in the approximate solution for the infiltration problem. However, the series has not yet been proven to converge. Tao [40] established convergence of a closely related series, firstly by verifying convergence at a particular time T_0 at the free boundary to the specified boundary value, in our notation $V(Y, T) = 1$, then by extending the series by comparison, to other values of Y and T. Even if a radius of convergence could be established, the series construction is not meant to compete with existing numerical methods in the efficient and accurate approximate solution of the water transport problem in unsaturated soil. Its main advantage is that it does provide an exact expression for the second and higher infiltration coefficients, admittedly only for a particular nonlinear transport model that has reasonable resemblance to some real soils [23].

There is a direct linear relationship between the location of the free boundary in the modified Stefan problem, and the cumulative infiltration in the problem of water transport in unsaturated soil. The second Philip infiltration coefficient is obtained explicitly as a function of the nonlinearity parameter C. This is the major specific output of this paper. The single continuous nonlinear parameter C connects the linear model ($C \to \infty$) to a nonlinear model in the Green–Ampt class ($C = 1$). Over this range, the second infiltration coefficient varies from $(K_s - K_n)/2$ to $(K_s - K_n)/3$. Surprisingly, the latter is only half that predicted by the standard Green–Ampt model but it is more realistic. The great majority of experimental results are within the range predicted by the current series construction.

In the soil hydrological model adopted here, for the benefit of mathematical simplicity, the assumption is made that $K'(\theta)/D(\theta)$ is constant. This is equivalent to setting the parameter γ in (1.7) to zero. We expect that the infiltration parameters calculated here would be indicative of those that apply for infiltration into soils for which $D(\theta)$ and $K'(\theta)$ are approximately proportional, as has commonly been observed [5,10].

Appendix: Recurrence relation for series coefficients

Assume that $\Sigma(T)$ and $V(Y, T)$ can be expanded as power series in $T^{1/2}$, of the form

$$\Sigma = \sum_{j=0}^{\infty} \gamma_j T^{j/2},$$

$$V = \mathrm{e}^{-Y^2/4} \sum_{j=0}^{\infty} C_j T^{j/2} \Psi\left(\frac{j+1}{2}, \frac{1}{2}, \frac{Y^2}{4}\right) = \sqrt{\pi} \sum_{j=0}^{\infty} C_j T^{j/2} \left\{ \frac{{}_1F_1\left(\frac{-j}{2}; \frac{1}{2}; \frac{-Y^2}{4}\right)}{\Gamma\left(1 + \frac{j}{2}\right)} - \frac{Y {}_1F_1\left(\frac{1-j}{2}; \frac{3}{2}; \frac{-Y^2}{4}\right)}{\Gamma\left(\frac{1+j}{2}\right)} \right\}. \tag{A.1}$$

The generalised hypergeometric functions ${}_pF_q$ are defined in [42, Sect. 9.14]. In order that the first term in (A.1) agrees with (3.11), it follows that

$$C_0 = [\sqrt{\pi}\,\mathrm{erfc}(\gamma_0/2)]^{-1}. \tag{A.2}$$

Formal differentiation of the power series term by term, gives

$$V_Y(Y, T) = -\sqrt{\pi} \sum_{j=0}^{\infty} C_j T^{j/2} \left\{ \frac{{}_1F_1\left(\frac{1-j}{2}; \frac{1}{2}; \frac{-Y^2}{4}\right)}{\Gamma\left(\frac{1+j}{2}\right)} - \frac{Y {}_1F_1\left(1 - \frac{j}{2}; \frac{3}{2}; \frac{-Y^2}{4}\right)}{\Gamma\left(\frac{j}{2}\right)} \right\}. \tag{A.3}$$

Now the boundary conditions at the free boundary $Y = \Sigma(T)$ are

$$V(\Sigma(T), T) = 1,$$
$$V_Y(\Sigma(T), T) = R\,T^{1/2} - \frac{C}{2}\sum_{j=0}^{\infty}(1+j)\gamma_j T^{j/2}. \tag{A.4}$$

In order to express all terms within these as power series, we use the identity (e.g. [43, Sect. 3.3])

$$\left(\sum_{m=0}^{\infty}\gamma_m x^m\right)^k = \sum_{n=0}^{\infty} A_n(k)x^n,$$
$$A_n(k) = \sum_{\mathbf{w}\in E_n}\frac{k!\gamma_1^{w_1}\gamma_2^{w_2}\cdots\gamma_n^{w_n}\gamma_0^{k-\Sigma_{i=1}^n w_i}}{w_1!w_2!\cdots w_n!\Gamma(1+k-\Sigma_{i=1}^n w_i)} \tag{A.5}$$

and E_n is a subset of n-tuples of the natural numbers,

$$E_n = \{\mathbf{w}\in\mathbb{N}^n : w_1 + 2w_2 + \cdots + nw_n = n\}. \tag{A.6}$$

Now from the power series for ${}_1F_1$ and (A.5), we deduce

$${}_1F_1\left(\alpha - \frac{j}{2}; \beta + \frac{1}{2}; \frac{-\Sigma^2(T)}{4}\right) = \sum_{i=0}^{\infty} T^{i/2}\sum_{\mathbf{w}\in E_i} G(i, j, \alpha, \beta; \mathbf{w}),$$
$$G(i, j, \alpha, \beta; \mathbf{w}) = \frac{\gamma_1^{w_1}\gamma_2^{w_2}\cdots\gamma_i^{w_i}}{w_1!w_2!\cdots w_i!}(-1)^q\gamma_0^{\zeta}\frac{\left(\alpha-\frac{j}{2}\right)_q\left(\frac{1}{2}\right)_q}{\left(\beta+\frac{1}{2}\right)_q}\times{}_2F_2\left(\alpha-\frac{j}{2}+q, \frac{1}{2}+q; \beta+\frac{1}{2}+q, \frac{1}{2}+\zeta; \frac{-\gamma_0^2}{4}\right),$$
$$q = \frac{1}{2}\left(\zeta + \sum_{k=1}^{i} w_k\right), \quad \zeta = \frac{1}{2} - \frac{1}{2}(-1)^{\sum_{k=1}^{i} w_k}. \tag{A.7}$$

The boundary conditions (A.4) may be written

$$1 = C_0 e^{-\frac{\gamma_0^2}{4}}\Psi\left(\frac{1}{2}, \frac{1}{2}, \frac{\gamma_0^2}{4}\right) + \sum_{j=1}^{\infty} T^{\frac{j}{2}}\left\{C_j e^{-\frac{\gamma_0^2}{4}}\Psi\left(\frac{1}{2}+\frac{j}{2}, \frac{1}{2}, \frac{\gamma_0^2}{4}\right) - C_0\gamma_j e^{-\frac{\gamma_0^2}{4}} + Q(0, j)\right\},$$
$$-\sqrt{C(C-1)}T^{\frac{1}{2}} + \frac{C}{2}\sum_{j=0}^{\infty}(1+j)\gamma_j T^{\frac{j}{2}} = C_0 e^{-\frac{\gamma_0^2}{4}}$$
$$+\sum_{j=1}^{\infty} T^{\frac{j}{2}}\left\{C_j e^{-\frac{\gamma_0^2}{4}}\Psi\left(\frac{j}{2}, \frac{1}{2}, \frac{\gamma_0^2}{4}\right) - \frac{C_0\gamma_0\gamma_j}{2} e^{-\frac{\gamma_0^2}{4}} + Q\left(\frac{1}{2}, j\right)\right\}; \tag{A.8}$$

where $Q(\alpha, j)$ is defined as

$$
\begin{aligned}
\frac{Q(\alpha, j)}{\sqrt{\pi}} = & -\sum_{p=1}^{j-1} \gamma_{j-p} \sum_{i=0}^{p} \frac{C_{p-i}}{\Gamma\left(\frac{1}{2}-\alpha+\frac{p-i}{2}\right)} \sum_{\mathbf{w} \in E_i} G\left(i, p-i, \alpha+\frac{1}{2}, 1; \mathbf{w}\right) \\
& + \frac{C_0}{\Gamma(1-\alpha)} \sum_{\mathbf{w} \in E_j, w_j \neq 1} G(j, 0, \alpha, 0; \mathbf{w}) \\
& + \sum_{i=1}^{j-1} \frac{C_{j-i}}{\Gamma\left(1-\alpha+\frac{j-i}{2}\right)} \sum_{\mathbf{w} \in E_i} G(i, j-i, \alpha, 0; \mathbf{w}) \\
& - \gamma_0 \sum_{i=1}^{j-1} \frac{C_{j-i}}{\Gamma\left(\frac{1}{2}-\alpha+\frac{j-i}{2}\right)} \sum_{\mathbf{w} \in E_i} G\left(i, j-i, \alpha+\frac{1}{2}, 1; \mathbf{w}\right) \\
& - \frac{C_0 \gamma_0}{\Gamma\left(\frac{1}{2}-\alpha\right)} \sum_{\mathbf{w} \in E_j, w_j \neq 1} G\left(j, 0, \alpha+\frac{1}{2}, 1; \mathbf{w}\right)
\end{aligned} \tag{A.9}
$$

Equating terms of order $T^{j/2}$ for $j \geq 2$, we can derive the explicit recurrence relations

$$
\begin{aligned}
C_j &= \frac{2\gamma_0 Q\left(\frac{1}{2}, j\right) - \left[2(j+1)+\gamma_0^2\right] Q(0, j)}{(j+1)(j+2)\mathrm{e}^{-\gamma_0^2/4} \Psi\left(\frac{3}{2}+\frac{j}{2}, \frac{1}{2}, \frac{\gamma_0^2}{4}\right)}, \\
\gamma_j &= \frac{4\Psi\left(\frac{1}{2}+\frac{j}{2}, \frac{1}{2}, \frac{\gamma_0^2}{4}\right) Q\left(\frac{1}{2}, j\right) - 4\Psi\left(\frac{j}{2}, \frac{1}{2}, \frac{\gamma_0^2}{4}\right) Q(0, j)}{(j+1)(j+2) C\, \Psi\left(\frac{3}{2}+\frac{j}{2}, \frac{1}{2}, \frac{\gamma_0^2}{4}\right)}.
\end{aligned} \tag{A.10}
$$

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